# On plane linear magnetohydrodynamic waves

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This paper investigates the perturbation modes of the steady parallel flow of a compressible fluid of finite constant viscosity and electrical conductivity in a uniform arbitrarily oriented magnetic field. In particular, in addition to the classical fast and slow sound and Alfvén modes, one obtains some waves whose amplitude is finite whenever the diffusive coefficients are finite and which resemble the diffusion of vorticity from stream surfaces in classical hydrodynamics. The paper also re-interprets upward-facing MHD waves and upstream wakes in a new way.

## 1. Introduction

The purpose of this study is to investigate the perturbation modes of the steady parallel flow of a compressible fluid of finite constant viscosity and electrical conductivity in a uniformly oriented magnetic field.

The equivalent problem for a non-conducting viscous gas was considered by Lagerstrom, Cole & Trilling (1949), who developed useful asymptotic techniques and showed how the perturbation motion may be decomposed uniquely into a vorticity-diffusion component and an irrotational diffusing wave component. It was then shown (e.g. by Trilling 1955) that the addition of the terms required to account for finite thermal conductivity introduces an additional entropy mode which is linearly independent of the vorticity and pressure wave modes when the Prandtl number is $\frac{3}{4}$.

The Oseen small disturbance motion of an incompressible steady viscous conducting fluid has been studied by Gourdine (1961), who constructed a fundamental solution made up of three linearly independent modes; while these do not separate the effects of pressure, viscosity and conductivity, their asymptotic expansion at large distances from the origin allows such separate combinations and makes it possible to construct three of the modes studied in this paper.

Other studies of the same problem of linearized incompressible real plasma flow were reported by Imai (1960) who stressed specific solutions at low magnetic Reynolds numbers where the magnetic force may be considered as given independently and by Busemann (1961) who presented a particular flow pattern for the case where the magnetic and hydrodynamic Reynolds numbers are equal and the magnetic pressure is equal to the dynamic pressure.

One of the points which are brought out clearly by Busemann and also by many other investigators (e.g. Sears & Tamada 1960 and Stewartson 1961) is the possibility that when the magnetic pressure exceeds the dynamic pressure, there exists a wake upstream of an obstacle and that therefore the conventional specification of boundary conditions for flow past an obstacle may need re-examination. The experiments available at present are not yet conclusive on this point, though there is some qualitative indication of a strong upstream influence (Liepmann 1961).

At the same time, many solutions for highly conducting incompressible and compressible plasma flows past thin bodies were presented by Sears, Resler, McCune and others at Cornell University (e.g. Resler & McCune 1960; Lary 1960), by Kogan (1959) in the USSR and others, and they indicate that in some flight régimes there must be waves slanted upstream in the flow pattern.

While this considerable body of literature increased the concrete knowledge available on some specified steady flow patterns, the propagation of waves in plasmas and also in more general anisotropic continuous media was the subject of numerous studies (see, for example, Cole & Lynn 1959; Whitham 1960; Staniukovich *et al.* 1956; Gogosov & Barmin 1961), which generalized our understanding of complex propagation phenomena and clarified the relationships between Alfvén waves and slow and fast MHD waves.

The purpose of the present investigation is to apply the apparatus of plane asymptotic wave-solution combinations to make a systematic qualitative survey of the effects of finite compressibility, viscosity and electrical conductivity on plasma flows. All of the modes and patterns found by Gourdine, Kogan and others are again obtained and some new general modes are also found. An attempt is made to interpret the upstream wake as a singular perturbation phenomenon associated with Alfvén waves in a dispersive medium. The nature of upstream-facing MHD waves is also considered from several points of view, not only as the result of the classical method of Huyghens constructions, but also as the solution of a quasi-wave problem which has a positive rather than a negative diffusion coefficient and is therefore consistent with the second law of thermodynamics. Finally, by gathering and exhibiting together all the modes possible in a compressible viscous electrically conducting fluid, this study may offer helpful leads to an analysis of the proper boundary conditions and stability criteria for the flow of such a fluid.

## 2. The equations of motion

The laws of physics, applied to a slightly disturbed electrically conducting compressible viscous fluid in rectilinear translation through a uniform magnetic field, take the following form in Cartesian co-ordinates $x_i$, using the Einstein summation convention.

Conservation of mass:
$$\frac{\partial s}{\partial t}+U_j\frac{\partial s}{\partial x_j}+\frac{\partial u_j}{\partial x_j}=0. \tag{1}$$

Conservation of momentum:
$$\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}+\frac{1}{M^2}\frac{\partial s}{\partial x_i}+B^2\cos\alpha_j\left(\frac{\partial b_j}{\partial x_i}-\frac{\partial b_i}{\partial x_j}\right)=\frac{1}{R}\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{1}{3}\frac{\partial^2 u_j}{\partial x_j\partial x_i}\right). \tag{2}$$

Maxwell's equation:
$$\frac{\partial b_i}{\partial t}+U_j\frac{\partial b_i}{\partial x_j}+\cos\alpha_i\frac{\partial u_j}{\partial x_j}-\cos\alpha_j\frac{\partial u_i}{\partial x_j}=\frac{1}{R_M}\frac{\partial^2 b_i}{\partial x_j\partial x_j}. \tag{3}$$

In equations (1) to (3) the position co-ordinates $x_i$ are divided by a reference length $L$ and time is divided by the reference time $L/U_\infty$, where $U_\infty$ is the undisturbed velocity. The dependent variables are defined as follows: $u_i$ is the velocity perturbation in the $x_i$-direction referred to the velocity $U_\infty$; $s$ is the condensation $(\rho-\rho_\infty)/\rho_\infty$, where $\rho_\infty$ is the density of the undisturbed fluid; $b_i$ is the magnetic field perturbation in the $x_i$-direction referred to the undisturbed field intensity $B_\infty$. In addition to the flow velocity $U_\infty$ whose direction cosines are designated by $U_i$ and the field intensity $B_\infty$ whose direction cosines are $\cos\alpha_i$, the following parameters are needed to describe the flow: $M = U_\infty/a_\infty$ is the Mach number, where $a_\infty$ is the reference speed of sound; $B^2 = B_\infty^2/\rho_\infty\mu_\infty U_\infty^2$ is the ratio of magnetic to dynamic pressure, where $\mu_\infty$ is the magnetic permeability of the fluid (assumed constant); $R = U_\infty L/\nu_\infty$ is the hydrodynamic Reynolds number, where $\nu_\infty$ is the kinematic viscosity coefficient of the fluid (assumed constant); $R_M = \mu_\infty \sigma U_\infty L$ is the magnetic Reynolds number, where $\sigma$ is the electrical conductivity of the fluid (assumed constant).

Two major physical assumptions are made in specifying equations (1) to (3). In the absence of heat conduction, the viscous and Joule dissipation is proportional to the square of small perturbation gradients and therefore the linearized flow is isentropic. Also, displacement currents are neglected.

Under these conditions, the seven equations (1) to (3) are partial differential equations with constant coefficients and have formal solutions

$$\psi(t, x_i) = \overline{\psi}(\omega, \lambda_i)\, e^{\omega t+\lambda_i x_i}, \tag{4}$$

where $\omega/i$ is a reduced frequency and $\lambda_i/i$ a wave-number component in the $i$-direction. Substitution of (4) into (1) and (3) gives

$$\bar{s} = -\lambda_j \bar{u}_j/(\omega+\lambda_k U_k) \tag{5}$$

and

$$\bar{b}_i = \frac{\bar{u}_i\lambda_j\cos\alpha_j - \bar{u}_j\lambda_j\cos\alpha_i}{\omega+\lambda_k U_k - R_M^{-1}\lambda_j\lambda_j}. \tag{6}$$

When (5) and (6) are substituted into (2), a single-vector equation for the velocity perturbation is obtained

$$\begin{aligned} M^2\bar{u}_i\left[(\omega+\lambda_j U_j)^2\left(1-\frac{\lambda_k\lambda_k}{R(\omega+\lambda_j U_j)}\right) - \frac{B^2\lambda_j\lambda_k\cos\alpha_j\cos\alpha_k}{\{1-\lambda_l\lambda_l R_M^{-1}(\omega+\lambda_m U_m)^{-1}\}}\right] \\ + \frac{M^2B^2\bar{u}_j\lambda_i\lambda_k\cos\alpha_j\cos\alpha_k}{\{1-\lambda_l\lambda_l R_M^{-1}(\omega+\lambda_k U_k)^{-1}\}} \\ - \bar{u}_j\lambda_j\left\{\lambda_i\left[\left(1+\frac{(\omega+\lambda_k U_k)\,M^2}{3R}\right)+\frac{M^2B^2}{\{1-\lambda_l\lambda_l R_M^{-1}(\omega+\lambda_k U_k)^{-1}\}}\right]\right. \\ \left. - \frac{M^2B^2\lambda_k\cos\alpha_i\cos\alpha_k}{1-\lambda_l\lambda_l R_M^{-1}(\omega+\lambda_k U_k)^{-1}}\right\} = 0. \end{aligned} \tag{7}$$

Equation (7) is the fundamental small disturbance equation, several special cases of which are studied below.

## 3. Some simple wave flows

The simplest example of (7) is the acoustic problem for which

$$1/R = 1/R_M = B = 0$$

so that (7) becomes

$$M^2\bar{u}_i(\omega+\lambda_j U_j)^2 - \bar{u}_j\lambda_j\lambda_i = 0. \tag{8}$$

The wave-numbers $\lambda_i$ are computed by requiring the determinant of the coefficients of $\overline{u}_i$ in (8) to vanish, i.e.

$$M^4(\omega+\lambda_j U_j)^4 [M^2(\omega+\lambda_j U_j)^2 - \lambda_k \lambda_k] = 0. \tag{9}$$

This gives the two familiar solutions of constant properties along a stream surface and propagation at the speed of sound with respect to fluid particles. The next example concerns viscous acoustic waves; $R$ is now finite; $B$ and $1/R_M$ still vanish. Then,

$$M^2\overline{u}_i(\omega+\lambda_k U_k)^2\left(1-\frac{\lambda_k\lambda_k}{R(\omega+\lambda_j U_j)}\right)-\overline{u}_j\lambda_j\lambda_i\left(1+\frac{(\omega+\lambda_k U_k)M^2}{3R}\right)=0 \tag{10}$$

and the wave-numbers $\lambda_i$ satisfy the following relations

$$M^4(\omega+\lambda_j U_j)^2[(\omega+\lambda_j U_j)-R^{-1}\lambda_k\lambda_k]^2$$
$$\times [M^2(\omega+\lambda_j U_j)^2-\lambda_k\lambda_k-\tfrac{4}{3}M^2R^{-1}\lambda_k\lambda_k(\omega+\lambda_j U_j)] = 0. \tag{11}$$

The first bracket represents the transverse mode of viscous diffusion of vorticity from stream surfaces, while the second represents a diffusing longitudinal acoustic wave. The factoring of (11) describes the splitting phenomenon discussed by, among others, Lagerstrom *et al.* (1949).

Another simple case is that of the plane longitudinal magnetohydrodynamic wave in an electrically and mechanically ideal fluid. If the $x_1$-axis is taken along the $U_\infty$-velocity direction, then $\lambda_2 = \lambda_3 = 0$ and $\cos\alpha_1 = \cos\theta$; $\cos\alpha_2 = \sin\theta$; $\cos\alpha_3 = 0$, where $\theta$ is the angle between the undisturbed field and the undisturbed velocity.

The equations of motion for that case are

$$\overline{u}_1[M^2(\omega+\lambda)^2-(1+M^2B^2\sin^2\theta)\lambda^2]+\overline{u}_2 M^2B^2\lambda^2\sin\theta\cos\theta = 0,$$
$$\overline{u}_1\lambda^2B^2\sin\theta\cos\theta+\overline{u}_2[(\omega+\lambda)^2-B^2\lambda^2\cos^2\theta] = 0, \tag{12}$$

which yield the roots

$$(\omega+\lambda)^2/\lambda^2 = [1+M^2B^2 \pm \{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}]/2M^2. \tag{13}$$

These are the fast and slow magnetohydrodynamic waves. In particular, if there is no convection velocity and $\omega$, $B$ are redefined in terms of the speed of sound as $\omega = \omega'/M$, $B = B'/M$, then (13) becomes

$$(\omega'/\lambda')^2 = c^2/a^2 = \tfrac{1}{2}[1+B'^2 \pm \{(1+B'^2)^2-4B'^2\cos^2\theta\}^{\frac{1}{2}}] \tag{13a}$$

from which the Huyghens wave-fronts for MHD waves may be constructed (see figure 1).

This Huyghens construction gives a convenient technique for studying plane flows of various régimes, as Resler & McCune (1960) and Kogan (1959) have done. It shows the wave-front arrangement and the flow patterns which would result from the superposition on it of a uniform velocity of arbitrary direction and magnitude. A simple geometrical construction shows that off the axis parallel to the $B$ field vector, four tangents to the wave front may be drawn from any point inside the cusped triangles or outside the oval, and two from any point inside the oval but outside the triangles. In fact the four tangents for an inside point F are shown on the sketch in figure 2; as F crosses the curve AC′, the tangents $T_3$, $T_4$ are lost; $T_1$ and $T_2$ are retained.

As F approaches the axis, the tangents $T_2$, $T_3$ go through AA′. Similarly, a point F′ in the two-wave region between the triangles and the oval has two tangents (and the flow whose velocity vector ends at F′ has two wave and two elliptic solutions) except when F′ goes to the axis. One may think of the axis

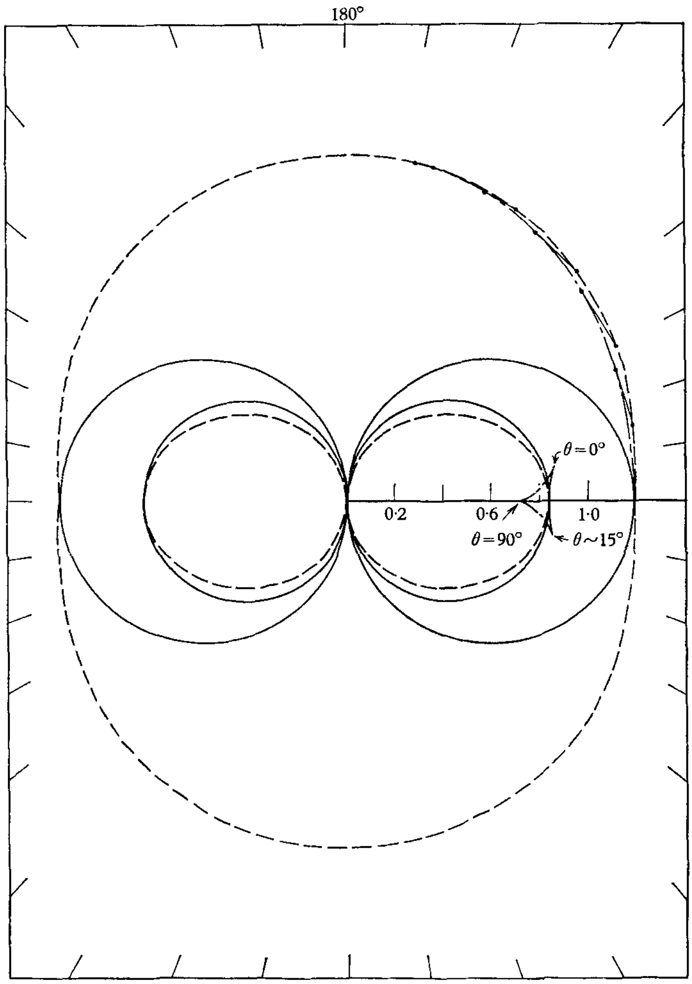


FIGURE 1. Variation with angle $\theta$ of phase and group velocities when the ratio of Alfvén velocity $u$ to sonic velocity $c$ is $2^{\frac{1}{2}}$ and $2^{-\frac{1}{2}}$.

itself as a wave-front (F′A; F′A′) which slides parallel to itself along lines of magnetic field; it allows no force discontinuity across it, but serves as a vortex sheet and a current sheet. A point on it travels along the field lines at a speed $ba/(a^2+b^2)^{\frac{1}{2}}$, where $b$ is the normal Alfvén speed and $a$ is the speed of sound.

To study these special transverse waves, one may consider the singular limiting process which occurs when the field $B$ makes a small but finite angle $\theta$ with $U_\infty$; the two roots of the characteristic equation disappear as $\sin\theta$ and take two boundary conditions with them.

In fact, equation (7) for two-dimensional steady problems with finite $\theta$ yields the roots

$$M^2\lambda_1^4 - (\lambda_1^2+\lambda_2^2)\,[(1+M^2B^2)\,\lambda_1^2 - B^2(\lambda_1\cos\theta+\lambda_2\sin\theta)^2] = 0. \tag{14}$$

This fourth-order equation for $\lambda_2$ (or $\lambda_1$) becomes quadratic for $\theta = 0$ (parallel field) where it takes the form

$$\xi^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^2 = \frac{(B^2-1)\,(1-M^2)}{1+B^2(M^2-1)}, \tag{15}$$

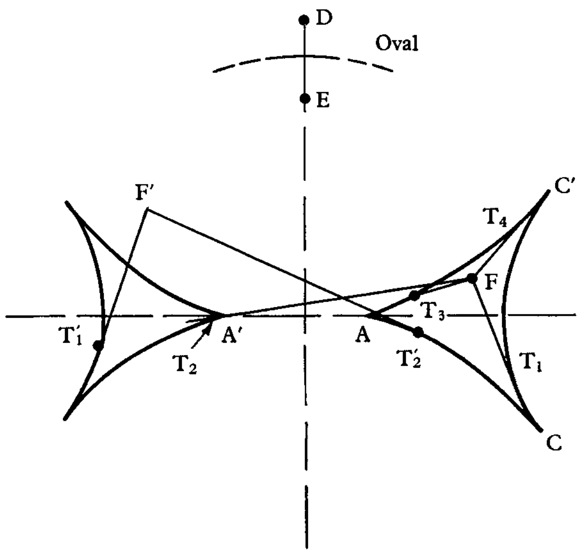


FIGURE 2. Huyghens construction for the inner (forward-facing) MHD waves when the field is not aligned with the stream.

and biquadratic for $\theta = \frac{1}{2}\pi$ (crossed field) where it yields

$$\xi^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^2 = \frac{1+B^2(M^2-1)\pm\{[1+B^2(M^2+1)]^2-4M^2B^2\}^{\frac{1}{2}}}{2B^2} \tag{16}$$

which has two positive roots, if $M^2 > 1/(1-B^2)$ (speed greater than $1+M^2B^2$, point D), and one positive root, if $M^2 < 1/(1-B^2)$ (point E). If $\sin\theta \ll 1$ then (14) takes the form

$$B^2\sin^2\theta\,\xi^4 + 2B^2\sin\theta\,\xi^3 + [B^2(1-M^2)-1]\,\xi^2 + 2\xi B^2\sin\theta + (M^2-1)\,(1-B^2) = 0, \tag{14a}$$

and has two sets of roots, one of which is given by (15) and the other is proportional to $\sec\theta$,

$$\xi_{1,2} = \pm\left(\frac{(B^2-1)\,(1-M^2)}{1+B^2(M^2-1)}\right)^{\frac{1}{2}}; \quad \xi_{3,4} = \sec\theta(-1\pm\{1+M^2B^2\}^{\frac{1}{2}}B^{-1}). \tag{17}$$

The $\xi_{3,4}$-solutions represent two sets of wave-fronts almost parallel to the $x_1$-axis, one of which always slopes upstream and, if $B^2 > 1/(1-M^2)$, the other also (in the elliptic region of Resler & McCune). The region of influence of a disturbance $\xi_{3,4}$ is a thin triangular hyperbolic boundary layer about the $x_1$-axis, which always faces upstream and sometimes also downstream. The boundary-layer thickness is $O(\sin\theta)$. If the total disturbance consists of a finite $u_2$-velocity disturbance (e.g. a thin body) then there are in this electric boundary layer finite $u_1$-, $b_1$-disturbances which in the limit of $\theta \to 0$ yield current and vortex sheets on the surface of the body and upstream of it but induce no force, and finite $u_2$-, $b_2$-disturbances associated with the $\xi_{1,2}$ roots.

A more general case of (7) is obtained if the velocity cosines $U_j$ are left arbitrary, but the x,-axis is aligned with the magnetic field $B$,. In that case, for the ideal fluid, $\cos\alpha_1 = 1$; $\cos\alpha_2 = \cos\alpha_3 = 0$; $1/R = 1/R_M = 0$ and the determinant of coefficients in equation (7) takes the form

$$\begin{vmatrix} M^2(\omega+\lambda_k U_k)^2-\lambda_1^2 & -\lambda_1\lambda_2 & -\lambda_1\lambda_3 \\ -\lambda_1\lambda_2 & M^2(\omega+\lambda_k U_k)^2-\lambda_2^2-M^2B^2(\lambda_1^2+\lambda_{2l}^2) & -\lambda_2\lambda_3(1+M^2B^2) \\ -\lambda_1\lambda_3 & -\lambda_2\lambda_3(1+M^2B^2) & M^2(\omega+\lambda_k U_k)^2-\lambda_3^2-M^2B^2(\lambda_1^2+\lambda_3^2) \end{vmatrix} = 0. \quad (18$$

After a certain amount of manipulation, it can be shown that (18) leads to the following relation for the wave-numbers $\lambda_i$:

$$[(\omega+\lambda_k U_k)^2-\lambda_1^2 B^2] \times \{M^2(\omega+\lambda_k U_k)^4-\lambda_k\lambda_k[(1+M^2B^2)(\omega+\lambda_k U_k)^2-\lambda_1^2 B^2]\} = 0, \quad (19)$$

which is the most general ideal-fluid perturbation equation because it is three-dimensional and allows an arbitrary angle between fluid flow and magnetic field. To generalize it formally, by allowing all $\cos\alpha_j$ to have non-vanishing values, involves a rotation of axes. Since all operators except $\lambda_1^2 B^2$ are independent of axis orientation, the formula is generalized simply by replacing $B^2\lambda_1^2$ by

$$B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j.$$

Equation (19) shows that the sixth-order system splits into a second-order equation which represents a pair of Alfvén waves and a fourth-order equation which represents a fast and a slow set of MHD waves. The Alfven waves are transverse waves which move at the appropriate velocity $B$ with respect to the fluid along the direction of the magnetic field lines; no $(u_1, s, b_1)$-discontinuities occur across them, but there may be disoontinuities in the components ($u_2$, us, $b_2$, $b_3$) which are tangent to the wave-front; in fact, this special solution may be found by inspection of the original system (1) to (3).

An observer who rides with the fluid and observes an Alfven wave sees a front which satisfies the relation

$$\lambda_1^2 - B^2(\lambda_1\cos\theta+\lambda_2\sin\theta)^2 = 0, \quad (19a)$$

or

$$\frac{\lambda_2}{\lambda_1} = \frac{-B\cos\theta \pm 1}{B\sin\theta}, \quad (19b)$$

where the minus sign applies to waves 'above' ($x_2 > 0$) the source. When $\cos\theta = 0$, these reduce to $\lambda_2/\lambda_1 = \pm$ 1/B which are the conventional Alfven waves propagated along normal field lines. When $B\cos\theta < -1$, the wave-front has a forward streamwise velocity component greater than the convection velocity. Therefore it propagates upstream as it moves upward (away from the moving observer). The slope of the characteristics decreases as $\sin\theta \to 0$ so that in the case of an almost parallel field, the Alfvén wave-train appears to become an upstream wake.

In plane flows there can be no Alfvén modes because (for $x$ parallel to the $B$ vector)

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \quad (u \equiv 0), \tag{19c}$$

so that one must have $\partial v/\partial y = 0$. Since far from the disturbance source ($y \to \infty$) perturbations must vanish, it follows that $v \equiv 0$. This result is similar to the result obtained for linearized (Prandtl–Glauert) transonic flow which cannot exist in two dimensions but gives Jones slender-body theory in three dimensions.

The fourth-order equation in (19) represents a combination of fast and slow waves; it is a generalization of (14) and carries a full complement of perturbations. Many special solutions of this set have been given in the literature. Note that if the field is aligned with or normal to the magnetic field, there are one or two pairs of waves (the equation becomes quadratic or biquadratic), while in the case of an arbitrary angle there are four separate roots which may be found from the Huyghens construction.

## 4. Plasma flows with finite conductivity and vanishing viscosity

When finite conductivity ($R_M$) is introduced into the problem, all the magnetic terms are multiplied by the factor $[1-\{\lambda_k\lambda_k/R_M(\omega+\lambda_k U_k)\}]^{-1}$ as can be seen from (6); (19) becomes

$$\begin{aligned}[(\omega+\lambda_k U_k)(\omega+\lambda_k U_k-\lambda_k\lambda_k R_M^{-1})-\lambda_1^2 B^2]& \\ \times\{M^2(\omega+\lambda_k U_k)^4(1-\{\lambda_k\lambda_k/R_M(\omega+\lambda_k U_k)\}& \\ -\lambda_m\lambda_m[(\omega+\lambda_k U_k)^2(1+M^2B^2-\{\lambda_k\lambda_k/R_M(\omega+\lambda_k U_k)\})-\lambda_1^2B^2]\} &= 0. \end{aligned} \tag{20}$$

The split into Alfvén waves and a pair of MHD waves is still found, but all the waves now have a diffusion pattern superposed on them.

For example, the Alfvén mode in a parallel field satisfies the equation

$$(\omega+\lambda_1)^2-\lambda_1^2B^2 = (\omega+\lambda_1)R_M^{-1}\lambda_j\lambda_j. \tag{21}$$

In particular, if the flow pattern is steady, (21) becomes

$$(1-B^2)\lambda_1 = \lambda_j\lambda_j/R_M, \tag{22}$$

which is almost identical with the vorticity diffusion component of the Oseen equations (see Lagerstrom *et al.* 1949). But while in the Oseen flow the coefficient of the first-order term is always positive, here its sign changes when $B^2$ goes through unity. This is due to the fact that a collapsed wave pattern rather than a wake is diffusing. The fundamental (source) solution of (22) in two dimensions is

$$\psi = \exp\{\tfrac{1}{2}(1-B^2)R_M x_1\}K_0[\tfrac{1}{2}(1-B^2)R_M r], \quad \text{where} \quad r = (x_1^2+x_2^2)^{\frac{1}{2}}, \tag{23a}$$

and in three dimensions,

$$\psi = r^{-1}\exp[\tfrac{1}{2}(1-B^2)R_M\{x_1 - r|1-B^2|(1-B^2)^{-1}\}], \quad r = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}. \tag{23b}$$

Therefore, if $1-B^2 > 0$, then the flow pattern includes a wake-like pattern downstream of the source which diffuses as $x_1^{-\frac{1}{2}}$, but, if $1-B^2 < 0$, the pattern is located upstream; such upstream propagation for axially symmetric geometries may have been observed in Prof. Liepmann's laboratory at the California Institute of Technology.

In the general case of a magnetic field at an arbitrary angle $\theta$ with the velocity vector, in steady plane flow, one finds that

$$\frac{\lambda_2}{\lambda_1} = \frac{-B^2 \sin\theta \cos\theta \pm \{B^2 \sin^2\theta + (\lambda_1/R_M)(1-B^2) - (\lambda_1/R_M)^2\}^{\frac{1}{2}}}{B^2 \sin^2\theta + (\lambda_1/R_M)}. \tag{24}$$

If $\sin\theta = 0$, $R_M \gg 1$, and this leads to the solution (23); but if the limiting process is carried out in the alternate order $1/R_M = 0$, $B \sin\theta \ll 1$, the answer is

$$\lambda_2/\lambda_1 = \sec\theta(-B \pm 1)/B, \tag{25}$$

which is a solution of the form discussed in connexion with equation (17). The parameter of physical significance is $R_M B^2 \sin^2\theta$, the ratio of the squares of the thicknesses of the current sheet and the electric diffusion layer. Its behaviour as $\sin\theta \to 0$, $R_M \to \infty$ determines the flow structure and the nature of the electric boundary layer.

If $\cos\theta = 0$ (crossed fields), then the solution

$$\frac{\lambda_2}{\lambda_1} = \pm\frac{1}{B}\left(\frac{1-(\lambda_1/R_M)}{1+(\lambda_1/B^2R_M)}\right)^{\frac{1}{2}} \tag{26}$$

represents a diffusing Alfvén quasi-wave of the LCT[1] type. Its asymptotic solution for large $R_M$, for the case of a step input, for example, is

$$\psi = \tfrac{1}{2}\psi_0\left[1-\operatorname{erf}\left(\frac{(x_1 - x_2 B^{-1})}{\{(B^2+1)\,x_1/R_M B^2\}^{\frac{1}{2}}}\right)\right]. \tag{27}$$

The MHD mode is similarly affected by a finite $R_M$. The formula (20) for MHD waves may be rewritten:

$$R_M^{-1}(\omega+\lambda_k U_k)(\lambda_j\lambda_j)^2 - [(\omega+\lambda_k U_k)^2(1+M^2B^2) - \lambda_1^2 B^2 + (M^2/R_M)(\omega+\lambda_k U_k)^3]\,\lambda_j\lambda_j + M^2(\omega+\lambda_k U_k)^4 = 0. \tag{28}$$

When $R_M \to \infty$, the asymptotic solutions are

$$(\omega+\lambda_k U_k)^2(1+M^2B^2) - \lambda_1^2 B^2 = (\omega+\lambda_k U_k)\,\lambda_j\lambda_j/R_M \tag{29a}$$

and

$$\frac{M^2(\omega+\lambda_k U_k)^4}{(1+M^2B^2)(\omega+\lambda_k U_k)^2 - B^2\lambda_1^2} - \lambda_j\lambda_j = \frac{\omega+\lambda_k U_k}{R_M}\,\frac{B^2(\omega+\lambda_k U_k)^2[M^2(\omega+\lambda_k U_k)^2-\lambda_1^2]}{[(1+M^2B^2)(\omega+\lambda_k U_k)^2 - B^2\lambda_1^2]^2}\,\lambda_j\lambda_j. \tag{29b}$$

Equation (29$a$) is similar to the damped Alfvén equation, but the propagation velocity of this mode is $B/(1+M^2B^2)^{\frac{1}{2}}$ instead of $B$. The same type of upstream or downstream wake pattern is possible as for Alfvén waves, but for parallel fields the critical line is $B^2 = 1/(1-M^2)$ instead of $B^2 = 1$. In fact, the solution (29$a$) represents the diffusion of the wave defined by $\xi_{3,4}$ in equation (17), namely, the vortex sheet and current sheet situated along lines of magnetic field which move at a velocity $B/(1+M^2B^2)^{\frac{1}{2}}$ (the velocity appropriate to the inner cusp A on the Huyghens wave construction). On the other hand, (29$b$) represents a damped MHD wave; its left-hand side represents an LCT[1] damping, whose intensity depends on the flow pattern; in particular, different Fourier components have different diffusion rates and waves propagated in different directions have different diffusion rates.

A special example of some possible interest is steady flow with a parallel field. In that case $(\omega+\lambda_k U_k)$ becomes $\lambda_1$ and (29$b$) becomes

$$\frac{(M^2-1)(1-B^2)}{1+B^2(M^2-1)}\frac{\partial^2\psi}{\partial x_1^2}-\left(\frac{\partial^2\psi}{\partial x_2^2}+\frac{\partial^2\psi}{\partial x_3^2}\right)=\frac{1}{R_M}\left[\frac{B^2(M^2-1)}{1-B^2(1-M^2)}\right]\frac{\partial}{\partial x_1}\frac{\partial^2\psi}{\partial x_i \partial x_i}, \quad (30)$$

where the partial differentials are introduced for the operators $\lambda_i$. This equation is similar to the LCT[1] equation for steady small disturbance supersonic viscous flow. An interesting feature of this equation is that, at first glance, one may think that for some combinations of $M$ and $B$ the diffusion coefficient on the right-hand side is negative; a more careful analysis shows that this is not so. Indeed, the flow consists of five regions.

(i) $M^2 < 1$; $B^2 < 1$: elliptic case; positive diffusion.

(ii) $M^2 < 1$; $1 < B^2 < 1/(1-M^2)$: hyperbolic case; the diffusion coefficient is negative; but this is the region of upstream facing waves and therefore $\partial/\partial x_1$ is negative, so that the diffusion is positive.

(iii) $M^2 < 1$; $1/(1-M^2) > B^2$: elliptic case with positive diffusion coefficient.

(iv) $M^2 > 1$; $B^2 < 1$: hyperbolic case; the diffusion coefficient is positive; the waves are swept downstream so that $\partial/\partial x_1$ is positive.

(v) $M^2 > 1$; $B^2 > 1$: elliptic case with positive diffusion coefficient.

The conclusion of this discussion, then, is that third-order damping always occurs in a manner similar to that predicted by the LCT[1] theory; but that, in order to insure this, it is necessary to have upstream propagation of disturbances in region (ii) as the Huyghens construction suggests. Note also the singular behaviour of the waves when $[1-B^2(1-M^2)]\to 0$ and the wave pattern merges into the upstream wake of (29$a$).

For unsteady one-dimensional waves, the basic equation is obtained from (28) by writing $\lambda_1=\lambda\cos\theta$, $\lambda_k U_k=\lambda$, $\lambda_k\lambda_k=\lambda^2$. The result is

$$\epsilon\lambda^4-[(1+M^2B^2)(\omega+\lambda)^2-\lambda^2B^2\cos^2\theta+\epsilon M^2(\omega+\lambda)^2]\lambda^2+M^2(\omega+\lambda)^4=0,$$
$$\epsilon=(\omega+\lambda)/R_M. \quad (31)$$

After some manipulation, the solution of (34) for small $\epsilon$ becomes

$$2M^2\left(\frac{\omega+\lambda}{\lambda}\right)^2=1+M^2B^2\pm\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}$$
$$+\epsilon M^2\left[1\pm\frac{1+M^2(B^2-2\cos^2\theta)}{\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}}\right]. \quad (32)$$

This formula consists of two parts; the non-diffusive part is identical with equation (13) and the part in square brackets gives the LCT[1] diffusion for fast and slow waves as a function of the angle between stream velocity and mean field. Replacing the operators $\omega$ and $\lambda$ by their values in terms of partial derivatives, one may rewrite (32) as

$$\frac{D^2\psi}{Dt^2}-\frac{1+M^2B^2\pm\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}}{2M^2}\frac{\partial^2\psi}{\partial x^2}$$
$$=\frac{1}{2R_M}\left[1\pm\frac{1+M^2(B^2-2\cos^2\theta)}{\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}}\right]\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \tag{33}$$

where $D/Dt$ is the substantial operator $(\partial/\partial t)+(\partial/\partial x)$. In the absence of convection, with sonic speed instead of convection speed as velocity reference, the generalization of (13$a$) becomes

$$\frac{\partial^2\psi}{\partial t'^2}-\frac{1+B'^2\pm\{(1+B'^2)^2-4B'^2\cos^2\theta\}^{\frac{1}{2}}}{2}\frac{\partial^2\psi}{\partial x^2}$$
$$=\frac{1}{2R'_M}\left[1\pm\frac{1+B'^2-2\cos^2\theta}{\{(1+B'^2)^2-4B'^2\cos^2\theta\}^{\frac{1}{2}}}\right]\frac{\partial}{\partial t'}\frac{\partial^2\psi}{\partial x^2}. \tag{33a}$$

While equations (33) and (33$a$) are quite general, several particular cases are of interest. When the fields are aligned, then $\cos\theta = 1$, and (33) becomes

$$\frac{D^2\psi}{Dt^2}-\frac{(1+M^2B^2)\pm(1-M^2B^2)}{2M^2}\frac{\partial^2\psi}{\partial x^2}=\frac{1}{2R_M}\left[1\pm\frac{M^2B^2-1}{1-M^2B^2}\right]\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \tag{34a}$$

or

$$\frac{D^2\psi}{Dt^2}-\frac{1}{M^2}\frac{\partial^2\psi}{\partial x^2}=0, \tag{34b}$$

$$\frac{D^2\psi}{Dt^2}-B^2\frac{\partial^2\psi}{\partial x^2}=\frac{1}{R_M}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}. \tag{34c}$$

These solutions represent a sound wave (34$b$) and a diffusing Alfvén wave (34$c$) as expected, since in the case of aligned fields the magnetic and acoustic components split and the acoustic problem is independent of electrical conductivity effects. In the crossed-field case, where $\cos\theta = 0$, one has

$$\frac{D^2\psi}{Dt^2}-\frac{1+M^2B^2}{M^2}\frac{\partial^2\psi}{\partial x^2}=\frac{1}{R_M}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \tag{35}$$

as one might have expected. Note also that the diffusion coefficient in (33$a$) is always positive because

$$(1+B'^2)^2-4B'^2\cos^2\theta-(1+B'^2-2\cos^2\theta)^2 = 4\sin^2\theta\cos^2\theta \geqslant 0, \tag{36}$$

the equal sign occurring in the limiting cases considered explicitly above. The same cannot be said of (33) where the sign of the quantity in brackets depends on the sign of $-1+M^2\cos^2\theta$. This merely suggests the presence of mixed waves propagating upstream in the manner discussed in connexion with equation (30), case (ii).

Returning to the solutions of (20) for low values of the magnetic Reynolds number $R_M$, the diffusing Alfvén mode becomes harmonic (Trilling & Kaplan 1961) while the MHD wave solutions become

$$\lambda_k\lambda_k = M^2(\omega+\lambda_k U_k)^2+R_M B^2\{(\omega+\lambda_k U_k)^2-\lambda_1^2\}/(\omega+\lambda_k U_k), \tag{37a}$$
$$\lambda_k\lambda_k = R_M(\omega+\lambda_k U_k). \tag{37b}$$

## 5. Plasma flows with finite viscosity and infinite conductivity

If one now considers finite viscosity ($R$) and no resistivity ($R_M \to \infty$) the fundamental equation (7) has two separate sets of solutions. The first is:

$$(\omega+\lambda_k U_k)(\omega+\lambda_k U_k - R^{-1}\lambda_j\lambda_j) - B^2\lambda_j\lambda_k \cos\alpha_j \cos\alpha_k = 0 \qquad (38a)$$

with

$$u_j\lambda_j = 0, \quad u_j \cos\alpha_j = 0. \qquad (38b)$$

This is identical with the damped Alfvén pattern of (20), with the viscous Reynolds number replacing the magnetic one. It gives the same shear waves and the same opportunity for upstream influence. The other wave system satisfies the equation

$$\lambda_k\lambda_k\left\{\left(1+M^2B^2+\frac{M^2(\omega+\lambda_k U_k)}{3R}\right)\left(1-\frac{\lambda_j\lambda_j}{R(\omega+\lambda_k U_k)}\right)(\omega+\lambda_k U_k)^2 - \left(1+\frac{(\omega+\lambda_k U_k)M^2}{3R}\right)B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j\right\}$$
$$= M^2(\omega+\lambda_k U_k)^4\left(1-\frac{\lambda_j\lambda_j}{R(\omega+\lambda_k U_k)}\right)^2. \qquad (39)$$

In particular, if $B = 0$, (39) reduces to the longitudinal component in (11). In general, if one follows the scheme of (28), one obtains a set of formal roots whose asymptotic expansions for $R \to \infty$ are (with the $x_1$ axis along $B$)

$$(\omega+\lambda_k U_k)^2 - \lambda_1^2 B^2(1+M^2B^2)^{-1} = (\omega+\lambda_k U_k)\lambda_j\lambda_j/R, \qquad (40a)$$

$$M^2(\omega+\lambda_k U_k)^4/\{(1+M^2B^2)(\omega+\lambda_k U_k)^2-\lambda_1^2B^2\}^{-1}\lambda_j\lambda_j = M^2(\omega+\lambda_k U_k)\alpha_R\lambda_j\lambda_j/R \qquad (40b)$$

with

$$\alpha_R \equiv \frac{(\omega+\lambda_k U_k)^2-\lambda_1^2B^2}{(1+M^2B^2)(\omega+\lambda_k U_k)^2-\lambda_1^2B^2}\left[\frac{1}{3}+\frac{(\omega+\lambda_k U_k)^2}{(1+M^2B^2)(\omega+\lambda_k U_k)^2-\lambda_1^2B^2}\right] + \frac{\lambda_1^2B^2[M^2(\omega+\lambda_k U_k)^2-\lambda_1^2]}{[(1+M^2B^2)(\omega+\lambda_k U_k)^2-\lambda_1^2B^2]^2}. \qquad (40c)$$

For plane steady flow, one obtains

$$\alpha_R = \frac{1-B^2}{1+(M^2-1)B^2}\left[\frac{1}{3}+\frac{1}{1+(M^2-1)B^2}\right]+\frac{B^2(M^2-1)}{[1+(M^2-1)B^2]^2},$$

which consists of two components: one is identical with magnetic diffusion and the other gives an additional effect peculiar to viscosity.

In the five field regions described earlier (see figure 3) one has the following effects:

(i) $M^2 < 1$, $B^2 < 1$; elliptic case, positive diffusion;

(ii) $M^2 < 1$, $1 < B^2 < 1/(1-M^2)$; hyperbolic case, forward-facing waves;

(iii) $M^2 < 1$, $1/(1-M^2) < B^2$; elliptic case, positive diffusion;

(iv) $M^2 > 1$, $B^2 < 1$; hyperbolic case, $\alpha_R > 0$, $\partial/\partial x_1 > 0$, rearward facing waves;

(v) $M^2 > 1$, $B^2 > 1$; elliptic case, positive diffusion.

The first of these patterns (40$a$) is similar to (29$a$), with the viscous Reynolds number $R$ replacing the magnetic Reynolds number $R_M$. The second pattern is again a diffusing MHD quasi-wave but the diffusion coefficient $\alpha_R$ has a rather complex form which reduces to the correct value $\frac{4}{3}$ in the absence of a magnetic field. As one might have expected, $\alpha_R$ is positive in region (iv) and negative in region (ii) of figure 3.

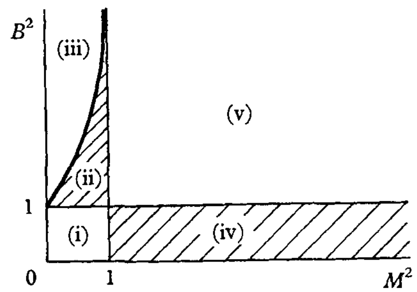


FIGURE 3. Mach number *vs* Alfvén number plot of regions where plane standing MHD waves are possible.

In the case of one-dimensional flow making an arbitrary angle $\theta$ with the undisturbed field direction the solution (40*b*) is equivalent to the differential equation

$$\frac{D^2\psi}{Dt^2}-\frac{1+M^2B^2\pm\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}}{2M^2}\frac{\partial^2\psi}{\partial x^2}$$
$$=\frac{1}{R}\left\{1+\frac{1}{6}\left[1\pm\frac{1+M^2B^2(1-2\cos^2\theta)}{\{(1+M^2B^2)^2-4M^2B^2\cos^2\theta\}^{\frac{1}{2}}}\right]\right\}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \quad (41)$$

where $D/Dt$ is the linearized substantial-derivative operator $(\partial/\partial t)+(\partial/\partial x)$. As in the case of electrical diffusion, the left-hand side gives the fast and slow MHD waves and the right-hand side gives the LCT[1] diffusion, showing in particular its dependence on orientation and on wave speed. For example, for the aligned field, one has

$$\frac{D^2\psi}{Dt^2}-\frac{1}{M^2}\frac{\partial^2\psi}{\partial x^2}=\frac{4}{3R}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \quad (42a)$$

$$\frac{D^2\psi}{Dt^2}-B^2\frac{\partial^2\psi}{\partial x^2}=\frac{1}{R}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \quad (42b)$$

and, for the crossed field, one has

$$\frac{D^2\psi}{Dt^2}-\frac{1+M^2B^2}{M^2}\frac{\partial^2\psi}{\partial x^2}=\frac{4}{3R}\frac{D}{Dt}\frac{\partial^2\psi}{\partial x^2}, \quad (42c)$$

which displays the longitudinal nature of acoustic and normal MHD waves (through the multiplier $\frac{4}{3}$) and the transverse nature of the Alfvén wave (42*b*).

The limiting case of small $R$ for Alfvén waves is similar to the case of small $R_M$ since the equations are identical. In the case of MHD waves, the asymptotic forms for small $R$ give

$$8\phi\Delta R^{-1}=(7\phi^2-B^2\lambda_1^2)\pm\{(7\phi^2-B^2\lambda_1^2)^2-48\phi^4\}^{\frac{1}{2}}, \quad (43)$$

where $\Delta=\lambda_k\lambda_k,\quad \phi=\omega+\lambda_k U_k,\quad \lambda_1^2=\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j.$

Note that when the magnetic field disappears, (43) gives the two roots

$$\phi = \tfrac{4}{3}R^{-1}\Delta, \quad \phi = R^{-1}\Delta, \tag{44 a, b}$$

which are the two LCT solutions for small Reynolds numbers and which show that, in their initial stages, the quasi-waves are diffusion patterns independent of Mach number.

In the case of steady plane flow with aligned field, (43) becomes

$$8\Delta R^{-1} = [(7-B^2) \pm (1-14B^2+B^4)^{\frac{1}{2}}]\lambda_1. \tag{45}$$

This equation has the same general structure as (22), but the structure of the diffusion constant is more complicated. When $7-\sqrt{48} < B^2 < 7+\sqrt{48}$, the diffusion constant is complex so that the wake is wavy and its phase structure depends on position. When $B^2 > 7$, the main part of the wake is directed upstream.

## 6. Plasma flows with finite conductivity and viscosity

This section considers the combined effects of viscosity and conductivity on an MHD flow. This involves a solution of the perturbation equation (7) with no terms omitted.

If the determinant of the coefficients of $\overline{u}_j$ is equated to zero, one obtains after some lengthy manipulation

$$\begin{aligned}[\phi^2(1-\epsilon_R)(1-\epsilon_M) - B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j] \\ \times \{M^2\phi^4(1-\epsilon_R)^2(1-\epsilon_M) - \lambda_k\lambda_k[\phi^2(1-\epsilon_R)[(1+\delta_R)(1-\epsilon_M)+M^2B^2] \\ - B^2\lambda_i\lambda_j(1+\delta_R)\cos\alpha_i\cos\alpha_j]\} = 0, \end{aligned} \tag{46}$$

where $\phi = \omega+\lambda_k U_k, \quad \epsilon_R = \lambda_k\lambda_k/\phi R, \quad \epsilon_M = \lambda_k\lambda_k/\phi R_M, \quad \delta_R = \tfrac{1}{3}\phi M^2/R.$

Equation (46) includes all the previous equations as special cases. It shows that general diffusive MHD flow perturbations are split into a pair of Alfvén transverse quasi-waves, independent of Mach number and always directed along lines of magnetic field, and into two pairs of MHD quasi-waves (a fast set and a slow set) which are coupled in a rather complex way and only become uncoupled for large values of $R$, $R_M$.

Consider first the set of Alfvén quasi-waves

$$\phi^2(1-\epsilon_R)(1-\epsilon_M) - B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j = 0. \tag{47}$$

This may be written as an equation for the Laplacian $\Delta = \lambda_k\lambda_k$ and as $R$ or $R_M \to \infty$ this reduces to (21) or (38*a*). If both $R$ and $R_M$ are finite, then one obtains

$$\Delta \equiv \lambda_k\lambda_k = \tfrac{1}{2}[\phi(R+R_M) \pm \{\phi^2(R-R_M)^2 + 4RR_MB^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j\}^{\frac{1}{2}}]; \tag{48}$$

when $B^2 \to 0$ the system has two simple roots

$$\Delta = R\phi \quad \text{and} \quad \Delta = R_M\phi. \tag{49a}$$

For steady flow with aligned field, (48) becomes

$$\Delta = \tfrac{1}{2}\lambda_1[R+R_M \pm \{(R-R_M)^2 + 4RR_MB^2\}^{\frac{1}{2}}], \tag{49b}$$

so that if $B^2 < 1$ there are two wakes which diffuse downstream and if $B^2 > 1$ there are two wakes, one going downstream and the other upstream, each wake being given by a superposition of sources of type (23) where $R_M(1-B^2)$ is replaced by the braced quantity in (49*b*).

Note that (47) is symmetric in $R$ and $R_M$ so that finite conductivity and viscosity affect Alfvén quasi-waves in the same way. If either $R$ (or $R_M$) goes to infinity while $R_M$ (or $R$) remains finite, then the two roots become

$$\Delta = R\lambda_1, \quad \Delta = R_M(1-B^2)\lambda_1 \quad \text{for} \quad R \gg R_M. \tag{50}$$

In the case of unsteady wave propagation, for large $R$ and $R_M$, one finds the following two roots

$$\Delta = \phi(R+R_M), \tag{51a}$$

$$\phi(R+R_M)R^{-1}R_M^{-1}\Delta = \phi^2 - B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j, \tag{51b}$$

which indicate that there is a diffusion from stream surfaces (51*a*) whose coefficient is the sum of the Reynolds numbers and an Alfvén quasi-wave (51*b*) whose diffusion coefficient is the harmonic mean of the magnetic and viscous Reynolds numbers.

When the Reynolds numbers $R$ and $R_M$ are small, then the asymptotic solutions take the form of simple diffusion patterns

$$\Delta = R\phi, \quad \Delta = R_M\phi. \tag{52}$$

The general pattern of diffusing Alfvén quasi-waves is therefore as follows. In the early stages, there is a superposition of viscous and electrical parabolic diffusion from stream surfaces of discontinuity. In the later stages ($R$, $R_M \to \infty$), there is the superposition of a diffusion pattern and an LCT-type Alfvén quasi-wave of transverse nature, moving along the magnetic field lines. There is no longer a real split between viscous and electrical modes; rather, both diffusion patterns are mixed and carry both electromagnetic and mechanical perturbation components.

The MHD modes satisfy the equation

$$\Delta\{\phi^2(1-\epsilon_R)[(1+\delta_R)(1-\epsilon_M)+M^2B^2] - (1+\delta_R)B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j\} - M^2\phi^4(1-\epsilon_R)^2(1-\epsilon_M) = 0 \tag{53}$$

which is the second bracket of (46). If the diffusion terms $\epsilon_M$, $\epsilon_R$, $\delta_R$ vanish, this is a fourth-order equation whose roots are the classical pairs of MHD fast and slow waves. One may expect that the introduction of diffusive higher-order terms will change the sharp (hyperbolic) waves to parabolic quasi-waves and add some diffusion modes. In the MHD modes, the viscosity and the resistivity do not play the symmetrical role which they play in the Alfvén modes, and the equation for $\Delta(\phi, \lambda_i, \lambda_j, \cos\alpha_i, \cos\alpha_j)$ is a cubic.

To find the asymptotic mode shapes one seeks roots of two forms: one set based on the sharp waves and the other on the magnetic field lines; thus we let

$$\Delta = \frac{M^2\phi^4}{(1+M^2B^2)\phi^2 - B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j}\left[1+M^2\phi\left(\frac{\alpha_R}{R}+\frac{\alpha_M}{R_M}\right)+\dots\right] \tag{54a}$$

and

$$\Delta = O(R, R_M). \tag{54b}$$

The first set of modes, substituted into (53), gives the solution

$$\frac{M^2(\omega+\lambda_k U_k)^4}{(1+M^2B^2)(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j}-\lambda_k\lambda_k = (\omega+\lambda_k U_k)\lambda_j\lambda_j$$
$$\times\left\{\frac{1}{R_M}\frac{B^2(\omega+\lambda_k U_k)^2[M^2(\omega+\lambda_k U_k)^2-\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j]}{[(1+M^2B^2)(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j]^2}\right.$$
$$+\frac{1}{R}\left[\frac{(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j}{(1+M^2B^2)(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j}\right.$$
$$\times\left(\frac{1}{3}+\frac{(\omega+\lambda_k U_k)^2}{(1+M^2B^2)(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j}\right)$$
$$\left.\left.+\frac{B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j[M^2(\omega+\lambda_k U_k)^2-\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j]}{[(1+M^2B^2)(\omega+\lambda_k U_k)^2-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j]^2}\right]\right\}, \tag{55}$$

which essentially states that the diffusion coefficient is the sum of the magnetic and viscous coefficients obtained in (29*b*) and (40).

The other two modes have the form

$$\Delta = \phi[(1+M^2B^2)R_M+R] \tag{56a}$$

and

$$\phi^2(1+M^2B^2)-B^2\lambda_i\lambda_j\cos\alpha_i\cos\alpha_j = \{(1+M^2B^2)R_M+R\}R^{-1}R_M^{-1}\phi\Delta. \tag{56b}$$

The root (56*a*) is a simple shear layer diffusion from streamlines whose diffusion coefficient, in distinction from (51*a*), depends on magnetic field strength; the root (56*b*) is a diffusion from the vortex and current sheet associated with the cusp A of the Huyghens diagram and its diffusion coefficient is the sum of those of (29) and (40*a*).

## 7. Conclusion

In general, when small viscous and electromagnetic diffusion is considered, there exist six sets of small plane disturbance modes in an MHD flow. These are an Alfvén wave, a fast and a slow MHD wave, and three modes for the diffusion of vorticity and electric current from stream surfaces.

The third-order coefficients of dispersion of the Alfvén and MHD fronts are direction-sensitive and take the correct limiting values as the fields become aligned, crossed or negligible.

Two vorticity and current diffusion modes have different first-order dispersion coefficients related to the flow parameters ($M$, $B$, $R_M$, $R$); the third mode is an LCT third-order mode which propagates along the field lines with the velocity appropriate for the inner cusp in the Huyghens construction.

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